METHOD OF RESOLUTION IN THE THEORY OF SHELLS

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An approximate method of solving certain problems of the theory of thin plates and shells is developed in the following; it is called here the "method of resolution".

The operators of the equations involved are resolved, by introducing so-called connecting functions, in such a manner that the knowledge of these functions may lead to a comparatively simple solution of the problem. The problem reduces to the determination of the functions indicated from the condition of equivalence of the original problem with the resolved one; this determination is carried out with the aid of the method of Ritz. Such a method of resolution permits the equations of the theory of thin shells, referred to the principal curvature lines of the middle surface, to be subdivided into two systems of ordinary differential equations, each referred to one of the two principal curvatures, respectively. The obtained equations are treated as equations for displacements and slopes of two groups of curved bars extending, respectively, along the two principal curvatures of the middle surface of the shell.

1. Essence of the method of resolution. In order to explain the method under consideration we shall discuss here a special case. Suppose a certain domain Ω and its boundary S are prescribed. It is required to find a definite function $u(P)$ of the point P, differentiable a necessary number of times and satisfying the differential equation

$$Au = f(P) \tag{1.1}$$

in the domain Ω and boundary conditions of the form

$$\Gamma_j u = g_j(P) \qquad (j = 1, \ldots, r) \tag{1.2}$$

along the boundary S. The notations used here are as follows: A represents a linear differential operator; $f(P)$ is a function prescribed in Ω; Γ_j are linear, generally speaking, differential operators; $g_j(P)$ are functions prescribed along S.

Assume that the operator A can be resolved into some simpler linear operators having a sum equal to A and that each of the operators can be inverted comparatively simply; assume, furthermore, that the corresponding resolution of the boundary conditions (1.2), related to the resolution of the operator A, is possible as well. In order to simplify the following presentation we assume $A = A_1 + A_2$, which in connection with (1.1) leads to

$$A_1 u = f_1(P), \qquad A_2 u = -f_1(P) + f(P) \tag{1.3}$$

The function $f_1(P)$, which we shall call the connecting function, must be determined from the condition of equivalence of the original and the resolved equations. This condition consists in the case under consideration of the equality of the functions $u(P)$, appearing in both equations (1.3).

For the determination of $f_1(P)$ we use the direct method. To this end we choose a complete sequence (ϕ_n) of linearly independent coordinate elements in the Hilbertian functional space $L_2(\Omega)$ and we prescribe an approximation for $f_1(P)$ in the form

$$f_{1n}(P) = a_1\varphi_1(P) + \ldots + a_n\varphi_n(P) \tag{1.4}$$

Then, inverting the operators A_1 and A_2 in (1.3), we obtain

$$u_{1n} = A_1^{-1} f_{1n}(P) = \sum_{k=1}^{n} a_k u_{1k}, \qquad u_{2n} = -A_2^{-1} f_{1n}(P) + A_2^{-1} f(P) = u_{20} - \sum_{k=1}^{n} a_k u_{2k} \tag{1.5}$$

where

$$u_{20} = A_2^{-1} f(P), \qquad u_{1k} = A_1^{-1}\varphi_k(P), \qquad u_{2k} = A_2^{-1}\varphi_k(P) \tag{1.6}$$

The function u_{1n} satisfies the part of the boundary conditions (1.2) which is associated with the operator A_1; the function u_{2n} satisfies the boundary conditions which are associated with the operator A_2. We introduce the notations

$$\psi_n = u_{1n} - u_{2n} = -u_{20} + \sum_{k=1}^{n} a_k v_k, \qquad v_k = u_{1k} + u_{2k} \tag{1.7}$$

We shall consider the quantity ψ_n as an error function of the approximate solution, while the norm of the element ψ_n, denoted by $\| \psi_n \|$, will be used as a measure of the error; an expedient procedure for determination of the constants a_k [1] is then based upon the condition that $\| \psi_n \|^2$ assumes its minimum value. We determine the metric of the Hilbertian space under consideration by the formula

$$\| \psi_n \| = \sqrt{[\psi_n, \psi_n]}$$

and the quantity $[\psi_n, \psi_n]$ by one of the following formulas:

$$(\psi_n, \psi_n), \qquad (A\psi_n, \psi_n), \qquad (A\psi_n, A\psi_n) \tag{1.8}$$

The use of the second scalar product (1.8) is possible when the operator A is positive. From the condition that $\| \psi_n \|^2$ assume its minimum value we then are able to derive a system of linear algebraic equations for the constants a_k:

$$\sum_{k=1}^{n} a_k [v_k, v_m] = [u_{20}, v_m] \qquad (m = 1, \ldots, n) \tag{1.9}$$

The system (1.9) always admits a solution if the elements $v_1, \ldots, v_n$ are linearly independent of each other. In turn, the mutual linear independence of $v_1, \ldots, v_n$ is a consequence of that of $\phi_1, \ldots, \phi_n$, provided that the operator A can be invertible. Indeed, if both parts of the equality

$$\sum_{k=1}^{n} c_k v_k = A_1^{-1}\Big(\sum_{k=1}^{n} c_k \varphi_k\Big) + A_2^{-1}\Big(\sum_{k=1}^{n} c_k \varphi_k\Big)$$

which follows from (1.7), are multiplied by $A_1 A_2$, leading to

$$A_1 A_2\Big(\sum_{k=1}^{n} c_k v_k\Big) = A\Big(\sum_{k=1}^{n} c_k \varphi_k\Big) \tag{1.10}$$

and if we assume that it is possible to find such non-simultaneously-vanishing constants $c_1, \ldots, c_n$ that

$$\sum_{k=1}^{n} c_k v_k = 0,$$

then from (1.10) we obtain

$$A\Big(\sum_{k=1}^{n} c_k \varphi_k\Big) = 0, \qquad \sum_{k=1}^{n} c_k \varphi_k = 0$$

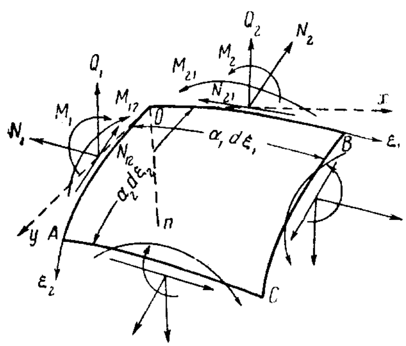

Fig. 1.

in contradiction to the assumption concerning mutual linear independence of $\phi_1, \ldots, \phi_n$.

The idea of the method of resolution can be realized in various ways, depending on the nature of the problem. Sometimes, for instance, it proves expedient to carry out an incomplete resolution, in the sense that not all of the deviced new operators have to be of simple structure and invertible in the process of solution. Several procedures of solution can be obtained in cases of the kind in question. Take, for example, the problem considered and assume that the operator A_1 is not to be inverted: if the function $f_1(P)$ is prescribed in such a way that the function u_{2n} satisfies all boundary conditions, then the coefficients a_k can be determined from the condition that u_{2n} satisfies approximately the first of Equations (1.3). Another possible way is to start directly from a function u_{1n} satisfying some of the boundary conditions associated with the operator A_1, to construct $f_1(P)$ and u_{2n}, and then to determine a_k from the condition of matching of u_{1n} and u_{2n}.

2. Resolution of the equilibrium equations for a shell element. Using the principal curvature lines ξ_1 and ξ_2 of the middle surface of the shell as coordinate lines (see Figure) and considering a line element of the shell extending in the direction of a ξ_1-line with the boundary surfaces ξ_2 = const and $\xi_2 + d\xi_2$ = const, we obtain the following system of equations for the equilibrium of such an element:

$$\frac{\partial \alpha_2 N_1}{\partial \xi_1} + N_{12}\frac{\partial \alpha_1}{\partial \xi_2} - Q_1\frac{\alpha_1\alpha_2}{R_1} = -\alpha_1\alpha_2(p_1 - q_1), \qquad \frac{\partial \alpha_2 M_{12}}{\partial \xi_1} + M_1\frac{\partial \alpha_1}{\partial \xi_2} = \alpha_1\alpha_2 m_2$$

$$\frac{\partial \alpha_2 N_{12}}{\partial \xi_1} - N_1\frac{\partial \alpha_1}{\partial \xi_2} = -\alpha_1\alpha_2(p_2 - q_2), \quad -\frac{\partial \alpha_2 M_1}{\partial \xi_1} + M_{12}\frac{\partial \alpha_1}{\partial \xi_2} + Q_1\alpha_1\alpha_2 = -\alpha_1\alpha_2 m_1$$

$$\frac{\partial \alpha_2 Q_1}{\partial \xi_1} + N_1\frac{\alpha_1\alpha_2}{R_1} = -\alpha_1\alpha_2(p_n - q_n), \qquad N_{12} + \frac{M_{12}}{R} = \tau \tag{2.1}$$

where a_1, a_2 are Lamé parameters, while R_1, R_2 are principal curvature radii of the middle surface of the shell; p_1, p_2, p_n, q_1, q_2, q_n, m_1, m_2 are forces and moments directed along x, y, n and referred to unit middle surface area; finally, the quantity τ is the intensity per unit length of opposite equal forces on OB and AC. The forces and moments q_1, q_2, q_n, τ, m_1, m_2 are external forces and moments with respect to the line element along ξ_1, but at the same time they represent the result of combined action of forces and moments which are inner forces and moments on OB and AC of the shell as a whole. On the other hand, the inner forces and moments acting on OA and BC will be external forces and moments with respect to the line element along ξ_2; their intensities are determined by the left-hand sides of Equations (2.1); therefore the equilibrium equations for a line element along a ξ_2-line assume the form

$$\frac{\partial \alpha_1 N_{21}}{\partial \xi_2} - N_2\frac{\partial \alpha_2}{\partial \xi_1} = -\alpha_1\alpha_2 q_1, \qquad -\frac{\partial \alpha_1 M_2}{\partial \xi_2} + M_{21}\frac{\partial \alpha_2}{\partial \xi_1} + Q_2\alpha_1\alpha_2 = -\alpha_1\alpha_2 m_2$$

$$\frac{\partial \alpha_1 N_2}{\partial \xi_2} + N_{21}\frac{\partial \alpha_2}{\partial \xi_1} - Q_2\frac{\alpha_1\alpha_2}{R_2} = -\alpha_1\alpha_2 q_2, \qquad \frac{\partial \alpha_1 M_{21}}{\partial \xi_2} + M_2\frac{\partial \alpha_2}{\partial \xi_1} = \alpha_1\alpha_2 m_1$$

$$\frac{\partial \alpha_1 Q_2}{\partial \xi_2} + N_2\frac{\alpha_1\alpha_2}{R_2} = -\alpha_1\alpha_2 q_n, \qquad N_{21} + \frac{M_{21}}{R_2} = \tau \tag{2.2}$$

Taking the algebraic sums of corresponding equations of the two systems (2.1) and (2.2), we obtain the equilibrium equations of an element of the shell as a whole. Thus, having introduced the connecting functions q_1, q_2, q_n, r, m_1, m_2 (we shall call them forces of inter - action), we were able to resolve the operators of the equilibrium equations in terms of partial derivatives into operators for separate coordinates.

For the sake of definiteness we shall use the version of the shell theory in Love's form [2], which is not compulsory for carrying out further considerations. Eliminating Q_1 and Q_2 in (2.1), (2.2) and omitting some terms within the limits of accepted degree of accuracy, we obtain

$$\begin{gathered}
\frac{\partial \alpha_2 N_1}{\partial \xi_1} + N_{12}\frac{\partial \alpha_1}{\partial \xi_2} - \frac{1}{R_1}\frac{\partial \alpha_2 M_1}{\partial \xi_1} = -\alpha_1\alpha_2(p_1 - q_1) \\
\frac{\partial \alpha_2 N_{12}}{\partial \xi_1} - N_1\frac{\partial \alpha_1}{\partial \xi_2} = -\alpha_1\alpha_2(p_2 - q_2) \\
\frac{\partial}{\partial \xi_1}\frac{\partial \alpha_2 M_1}{\alpha_1\,\partial \xi_1} - \frac{\partial}{\partial \xi_1}\left(\frac{M_{12}}{\alpha_1}\frac{\partial \alpha_1}{\partial \xi_2}\right) + N_1\frac{\alpha_1\alpha_2}{R_1} = -\alpha_1\alpha_2(p_n - q_n) + \frac{\partial \alpha_2 m_1}{\partial \xi_1} \\
\frac{\partial \alpha_2 M_{12}}{\partial \xi_1} + M_1\frac{\partial \alpha_1}{\partial \xi_1} = \alpha_1\alpha_2 m_2
\end{gathered} \tag{2.3}$$

and

$$\begin{gathered}
\frac{\partial \alpha_1 N_{21}}{\partial \xi_2} - N_2\frac{\partial \alpha_2}{\partial \xi_1} = -\alpha_1\alpha_2 q_1 \\
\frac{\partial \alpha_1 N_2}{\partial \xi_2} + N_{21}\frac{\partial \alpha_2}{\partial \xi_1} - \frac{1}{R_2}\frac{\partial \alpha_1 M_2}{\partial \xi_2} = -\alpha_1\alpha_2 q_2 \\
\frac{\partial}{\partial \xi_2}\frac{\partial \alpha_1 M_2}{\alpha_2\partial \xi_2} - \frac{\partial}{\partial \xi_2}\left(\frac{M_{21}}{\alpha_2}\frac{\partial \alpha_2}{\partial \xi_1}\right) + N_2\frac{\alpha_1\alpha_2}{R_2} = -\alpha_1\alpha_2 q_n + \frac{\partial \alpha_1 m_2}{\partial \xi_2} \\
\frac{\partial \alpha_1 M_{21}}{\partial \xi_2} + M_2\frac{\partial \alpha_2}{\partial \xi_1} = \alpha_1\alpha_2 m_1
\end{gathered} \tag{2.4}$$

We have omitted here also the last equations of the systems (2.1), (2.2), which with the assumptions introduced are equivalent to the condition $N_{12} = N_{21}$.

The equations obtained can be considered as equilibrium equations for the separate curved bars extending along the ξ_1- and ξ_2-lines and of width proportional to α_2 and α_1, respectively. Along their lateral surfaces such bars are acted upon by forces and moments of intensities N_{21}, M_{21} and N_{12}, M_{12}, which permit the moment equilibrium condition to be satisfied with respect to the n-axis in the absence of bending with respect to the same axis.

3. Resolution of the equations of the theory of shells. We shall adhere to the principle that the resolved equations have to represent two systems of equations for the displacements and slope angles of the two groups of line elements extending along ξ_1 and ξ_2, respectively. The equations connecting the stress resultants with the strain components and the strain components with the displacements shall be written in the form [2]

$$N_i = B(\varepsilon_i + \nu\varepsilon_k), \qquad N_{12} = N_{21} = L\gamma \qquad \left(B = \frac{Eh}{1-\nu^2},\ L = \frac{Eh}{2(1+\nu)}\right)$$

$$M_i = -D\ (\chi_i + \nu\chi_k), \qquad M_{12} = M_{21} = D(1-\nu)\chi \qquad \left(D = \frac{Eh^3}{12(1-\nu^2)}\right) \tag{3.1}$$

$$\varepsilon_i = \frac{1}{\alpha_i}\frac{\partial u_i}{\partial \xi_i} + \frac{u_k}{\alpha_i\alpha_k}\frac{\partial \alpha_i}{\partial \xi_k} - \frac{w}{R_i}, \qquad \chi_i = \frac{1}{\alpha_i}\frac{\partial \vartheta_i}{\partial \xi_i} + \frac{1}{\alpha_i\alpha_k}\frac{\partial \alpha_i}{\partial \xi_k}\vartheta_k$$

$$\gamma = -(n_1 + n_2), \qquad \chi = -(l_1 + l_2), \qquad \vartheta_i = \frac{u_i}{R_i} + \frac{1}{\alpha_i}\frac{\partial w}{\partial \xi_i} \qquad \binom{i=1,2}{k=2,1}$$

where

$$n_i = -L\left(\frac{1}{\alpha_k}\frac{\partial u_i}{\partial \xi_k} - \frac{1}{\alpha_i\alpha_k}\frac{\partial \alpha_k}{\partial \xi_i}u_k\right) = -L\beta_i \qquad \binom{i=1,2}{k=2,1}$$

$$l_i = -\frac{D(1-\nu)}{2}\left(\frac{1}{\alpha_k}\frac{\partial \vartheta_i}{\partial \xi_k} - \frac{1}{\alpha_i\alpha_k}\frac{\partial \alpha_k}{\partial \xi_i}\vartheta_k\right) = -\frac{D(1-\nu)}{2}\varphi_i \tag{3.2}$$

while E is the modulus of elasticity, ν is Poisson's ratio, h is the shell thickness, u_1, u_2, w are projections of displacement on the x-, y-, n-axes, respectively. Assume first $\nu = 0$; then, expressing the stress resultants in terms of displacements with the aid of Formulas (3.1), (3.2) and substituting into Equations (2.3), (2.4), we obtain the following two systems of equations for the two groups of line elements extending in the directions ξ_1 and ξ_2, respectively:

$$F_{11}[u_1^{(1)}, u_2^{(1)}, w^{(1)}, \vartheta_2^{(1)}] = -\alpha_1\alpha_2(p_1 - q_1) + \frac{\partial \alpha_1}{\partial \xi_2}n_1$$

$$F_{12}[u_1^{(1)}, u_2^{(1)}, w^{(1)}, \vartheta_2^{(1)}] = -\alpha_1\alpha_2(p_2 - q_2) + \frac{\partial \alpha_2 n_1}{\partial \xi_1} \tag{3.3}$$

$$F_{13}[u_1^{(1)}, u_2^{(1)}, w^{(1)}, \vartheta_2^{(1)}] = -\alpha_1\alpha_2(p_n - q_n) + \frac{\partial \alpha_2 m_1}{\partial \xi_1} - \frac{\partial}{\partial \xi_1}\left(\frac{l_1}{\alpha_1}\frac{\partial \alpha_1}{\partial \xi_2}\right)$$

$$F_{14}[u_1^{(1)}, u_2^{(1)}, w^{(1)}, \vartheta_2^{(1)}] = \alpha_1\alpha_2 m_2 + \frac{\partial \alpha_2 l_1}{\partial \xi_1}, \qquad \beta_1^{(1)} = -\frac{n_1}{L}, \qquad \varphi_1^{(1)} = -\frac{2}{D}l_1$$

and

$$F_{21}[u_1^{(2)}, u_2^{(2)}, w^{(2)}, \vartheta_1^{(2)}] = -\alpha_1\alpha_2 q_1 + \frac{\partial \alpha_1 n_2}{\partial \xi_2}$$

$$F_{22}[u_1^{(2)}, u_2^{(2)}, w^{(2)}, \vartheta_1^{(2)}] = -\alpha_1\alpha_2 q_2 + \frac{\partial \alpha_2}{\partial \xi_1}n_2 \tag{3.4}$$

$$F_{23}[u_1^{(2)}, u_2^{(2)}, w^{(2)}, \vartheta_1^{(2)}] = -\alpha_1\alpha_2 q_n + \frac{\partial \alpha_1 m_2}{\partial \xi_2} - \frac{\partial}{\partial \xi_2}\left(\frac{l_2}{\alpha_2}\frac{\partial \alpha_2}{\partial \xi_1}\right)$$

$$F_{24}[u_1^{(2)}, u_2^{(2)}, w^{(2)}, \vartheta_1^{(2)}] = \alpha_1\alpha_2 m_1 + \frac{\partial \alpha_1 l_2}{\partial \xi_2}, \qquad \beta_2^{(2)} = -\frac{n_2}{L}, \qquad \varphi_2^{(2)} = -\frac{2}{D}l_2$$

We find on the left-hand sides of Equations (3.3) and (3.4) the quantities n_2, l_2, ϑ_1 and n_1, l_1, ϑ_2, respectively, expressed by u_1, u_2, w and by ϑ_2 or ϑ_1. It is, furthermore, easy to conclude that the $F_{1j}(j = 1, 2, 3, 4)$ represent ordinary differential operators with respect to ξ_1, while the $F_{2j}(j = 1, 2, 3, 4)$ are such with respect to ξ_2. Equations (3.3) and (3.4) can be treated as equations for displacements and slope angles, identified by corresponding subscripts, for the two groups of line elements directed along ξ_1 and ξ_2. The quantities ϑ_2, ϑ_1 characterize the rotation of the cross-section of the element, β_1, β_2 indicate the shear along the element, ϕ_1, ϕ_2 give the shear across the element.

In order to achieve a complete resolution of the equations of the theory of shells in the sense indicated above, four interaction terms, namely n_1, n_2, l_1, l_2, had to be introduced additionally; they can be treated as intensities of oppositely equal forces and moments applied to the lateral faces of the elements under consideration and combined, respectively, to couples and bi-couples. There are altogether nine terms of the kind just mentioned in (3.3), (3.4). Nine matching conditions are necessary for the determination of those terms, and these conditions make (3.3), (3.4) equivalent to the equations of the theory of shells. Such conditions are

$$\begin{aligned} &u_1^{(1)} = u_1^{(2)}, \qquad \vartheta_i^{(i)} = \frac{u_k^{(k)}}{R_k} + \frac{1}{\alpha_k}\frac{\partial w^{(k)}}{\partial \xi_k} \\ &u_2^{(1)} = u_2^{(2)}, \qquad \beta_i^{(i)} = \frac{1}{\alpha_k}\frac{\partial u_i^{(k)}}{\partial \xi_k} - \frac{1}{\alpha_i\alpha_k}\frac{\partial \alpha_k}{\partial \xi_i}u_k^{(k)} \qquad \begin{pmatrix} i=1,2 \\ k=2,1 \end{pmatrix} \\ &w^{(1)} = w^{(2)}, \qquad \varphi_i^{(i)} = \frac{1}{\alpha_k}\frac{\partial \vartheta_i^{(k)}}{\partial \xi_k} - \frac{1}{\alpha_i\alpha_k}\frac{\partial \alpha_k}{\partial \xi_i}\vartheta_k^{(k)} \end{aligned} \tag{3.5}$$

If expressions for the interaction forces and moments are given with indefinite coefficients, then in solving (3.3), (3.4) these coefficients can be chosen in such a manner as to have the conditions (3.5) satisfied in some sense. Thus, the solution of the problem considered will be constructed by a direct method. It is usually convenient, and for a majority of problems acceptable, to have the operators (3.3), (3.4) inverted along separate lines ξ_2 = const, ξ_1 = const, and (3.5) satisfied at separate points. In this procedure methods of structural mechanics can be used for derivation of the solution of (3.3), (3.4), taking into account some specific properties of (3.3), (3.4) such as absence of bending with respect to the n-axis, torsion with a rigidity proportional to the moment of inertia, presence of the last equations of the systems (3.3), (3.4), etc.

If, in the process of resolution of the original equations, the only intention is to arrive at ordinary differential equations without having in mind the possibility of making use of the methods of structural mechanics, then the number of interaction forces and moments in (3.3), (3.4) can be reduced. Simple transformations permit the conclusion that

$$l_i = \frac{D(1-\nu)}{2LR_i}\, n_i + f_i\,[u_1^{(i)},\ u_2^{(i)},\ w^{(i)},\ \vartheta_k^{(i)}] \qquad \binom{i=2,1}{k=2,1} \tag{3.6}$$

where the f_i represent ordinary differential operators with respect to ξ_1 and ξ_2, respectively, for the quantities within the brackets. Making use of (3.6) we can eliminate l_1 and l_2 from (3.3), (3.4); this means elimination of the last equations in the systems (3.3), (3.4) and of the last two conditions of the system (3.5).

In the general case, when $\nu \neq 0$, the right-hand sides of the first four equations in (3.3), (3.4) include additional interaction forces and moments determined by the quantities $N_{1\nu}$, $M_{1\nu}$ in (3.3) and $N_{2\nu}$, $M_{2\nu}$ in (3.4) by means of the formulas

$$\begin{gathered} -\nu\left(\frac{\partial\alpha_k N_{i\nu}}{\partial\xi_i} - \frac{1}{R_i}\frac{\partial\alpha_k M_{i\nu}}{\partial\xi_i}\right), \qquad \nu\frac{\partial\alpha_i}{\partial\xi_k}N_{i\nu} \\ -\nu\left(\frac{\alpha_i\alpha_k}{R_i}N_{i\nu} + \frac{\partial}{\partial\xi_i}\frac{\partial\alpha_k M_{i\nu}}{\alpha_i\partial\xi_i}\right), \qquad -\nu\frac{\partial\alpha_i}{\partial\xi_k}M_{i\nu} \end{gathered} \quad \begin{pmatrix} N_{i\nu} = B\varepsilon_k^{(i)} \\ M_{i\nu} = -D\chi_k^{(i)} \end{pmatrix} \quad \binom{i=1,2}{k=2,1} \tag{3.7}$$

These quantities characterize relative extension across an element and relative change of angle between its lateral surfaces. For their determination four additional matching conditions will be necessary in the general case:

$$\varepsilon_i^{(k)} = \frac{1}{\alpha_i}\frac{\partial u_i^{(i)}}{\partial\xi_i} + \frac{u_k^{(i)}}{\alpha_i\alpha_k}\frac{\partial\alpha_i}{\partial\xi_k} - \frac{w^{(i)}}{R^i}, \quad \chi_i^{(k)} = \frac{1}{\alpha_i}\frac{\partial\vartheta_i^{(i)}}{\partial\xi_i} + \frac{1}{\alpha_i\alpha_k}\frac{\partial\alpha_i}{\partial\xi_k}\vartheta_k^{(i)} \qquad \binom{i=1,2}{k=2,1} \tag{3.8}$$

Resolution of the equations of the theory of shells into the systems (3.3), (3.4) is accompanied by resolution of the boundary conditions. The system (3.3) is to be supplemented by boundary conditions on the sides ξ_2 = const, the system (3.4) by boundary conditions on the sides ξ_1 = const of the bounding contour. A specific scheme of support will arise in this case when the line elements, separated from the shell, are not in equilibrium under the action of the applied loads and support reactions. It then becomes necessary to prescribe for one end of the element the slope angle and three displacements as indefinite parameters. As a result, the quantities indicated will be determined from the equilibrium conditions of the line element considered.

Thus far we have considered the problem of the theory of shells in terms of displacements. If the problem is formulated in terms of stress resultants, it may prove expedient to use an incomplete resolution of the original equations, leaving the continuity equations for the strain components of the middle surface of the shell [3] unresolved. Then the five interaction forces and moments, appearing in (2.3), (2.4), have to be determined starting from an approximate fulfilment of the conditions $N_{12} = N_{21}$, $M_{12} = M_{21}$ and of the three continuity equations for the strain components [3]; in addition, they have to satisfy all boundary conditions.

4. Examples. *1. Plate in bending.* Assuming $a_1 = a_2 = 1$, $R_1 = R_2 = \infty$, $u_1 = u_2 = 0$, $\xi_1 = x$, $\xi_2 = y$, we obtain from (3.3), (3.4) the following system of equations for the two groups of rectilinear elements directed along x and y:

$$D\frac{\partial^4 w^{(1)}}{\partial x^4} = p_n - q_n - \frac{\partial m_1}{\partial x}, \qquad D\frac{\partial^2 \vartheta_2^{(1)}}{\partial x^2} = m_2$$
$$D\frac{\partial^4 w^{(2)}}{\partial y^4} = q_n - \frac{\partial m^{(2)}}{\partial y}, \qquad D\frac{\partial^2 \vartheta_1^{(2)}}{\partial y^2} = m_1 \tag{4.1}$$

where $n_1 = n_2 = 0$, while l_1, l_2 are eliminated on the basis of (3.6). The number of interactions in (4.1) can be reduced to two, since from the second equations of (4.1) it follows that $\partial m_2/\partial y = \partial m_1/\partial x$. The matching conditions for (4.1) will be

$$w^{(1)} = w^{(2)}, \qquad \vartheta_2^{(1)} = \frac{\partial w^{(2)}}{\partial y} \qquad \text{or} \qquad \vartheta_1^{(2)} = \frac{\partial w^{(1)}}{\partial x} \tag{4.2}$$

2. Circular cylindrical shell. Assuming $a_1 = 1$, $a_2 = a$, $R_1 = \infty$, $R_2 = a$, $\xi_1 = x$, $\xi_2 = \psi$, $u_1 = u$, $u_2 = v$, we obtain from (3.3), (3.4)

$$B\frac{\partial^2 u^{(1)}}{\partial x^2} = -(p_1 - q_1), \qquad L\frac{\partial^2 u^{(2)}}{a^2\partial\psi^2} = -q_1 + \frac{\partial n_2}{a\partial\psi}$$
$$L\frac{\partial^2 v^{(1)}}{\partial x^2} = -(p_2 - q_2) + \frac{\partial n_1}{\partial x}, \qquad \frac{B}{a^2}\left(\frac{\partial^2 v^{(2)}}{\partial\psi^2} - \frac{\partial w^{(2)}}{\partial\psi}\right) + \frac{D}{a^4}\left(\frac{\partial^2 v^{(2)}}{\partial\psi^2} + \frac{\partial^3 w^{(2)}}{\partial\psi^3}\right) = -q_2,$$
$$D\frac{\partial^4 w^{(1)}}{\partial x^4} = (p_n - q_n) - \frac{\partial m_1}{\partial x}, \quad \frac{D}{a_4}\left(\frac{\partial^3 v^{(2)}}{\partial\psi^3} + \frac{\partial^4 w^{(2)}}{\partial\psi^4}\right) - \frac{B}{a}\left(\frac{\partial v^{(2)}}{a\partial\psi} - \frac{w^{(2)}}{a}\right) = q_n - \frac{\partial m_2}{a\partial\psi} \tag{4.3}$$
$$\frac{D}{2}\frac{\partial^2 \vartheta_2^{(1)}}{\partial x^2} = m_2 + \frac{\partial l_1}{\partial x}, \qquad \frac{D}{2}\frac{\partial^2 \vartheta_1^{(2)}}{a^2\partial\psi^2} = m_1 + \frac{\partial l_2}{a\partial\psi}$$
$$L\frac{\partial u^{(1)}}{a\partial\psi} = -n_1, \qquad L\frac{\partial v^{(2)}}{\partial x} = -n_2,$$
$$\frac{D}{2}\frac{\partial \vartheta_1^{(1)}}{a\partial\psi} = -l_1, \qquad \frac{D}{2}\frac{\partial \vartheta_2^{(2)}}{\partial x} = -l_2$$

It follows also that $\partial m_2/a\partial\psi = \partial m_1/\partial x$, and it is sufficient to satisfy one of the conditions (3.5) with respect to ϑ_1, ϑ_2.

A study of (4.3) and of the assumptions representing the basis for the analysis of arch dams with the aid of arch cantilevers [4] leads without difficulty to the conclusion that this kind of analysis reduces, if the variation in the thickness of the dam is disregarded, to the solution of (4.3) in connection with carrying out of the corresponding matching at $u_1 = q_1 = n_1 = n_2 = 0$ with l_1, l_2 eliminated. Indeed, the method of arch cantilevers is based upon subdivision of the dam into a system of vertical cantilevers and horizontal circular arches. These elements are subjected to external loads and internal interaction forces and moments in such a way as to make their radial and tangential displacements and their slope angles coincide. If $u_1 = q_1 = n_1 = n_2 = 0$, then the relations (4.3) become equations for the same displacements and slope angles of the cantilevers and arches, while Equations (3.5) become matching conditions and q_2, q_n, m_1, m_2 correspond to the internal interaction forces and moments of the method of arch cantilever.

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